Jacobi , Newton-Raphson.

① A, b θέλουμε $Ax = b \to x^{(k)}, \; k = 1, \ldots$

$$x^{(k)} = G\,x^{(k-1)} + M^{-1}b \quad \text{όπου} \quad G = M^{-1}N, \quad A = M - N$$

$$A = D + (A - D) = \overset{M}{\overset{\shortparallel}{D}} - \underbrace{(D - A)}_{N} \qquad M = D = \begin{bmatrix} \alpha_{11} & & \mathbb{O} \\ & \alpha_{22} & \\ \mathbb{O} & & \ddots \; \alpha_{nn} \end{bmatrix}$$

$$\exists\, M^{-1} \text{ αν } \alpha_{ii} \neq 0 \;\; \forall i \qquad M^{-1} = \begin{bmatrix} 1/\alpha_{11} & & \mathbb{O} \\ \mathbb{O} & \ddots & 1/\alpha_{nn} \end{bmatrix}$$

Εργασία.

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 1 \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 4 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 0 \\ -3 \\ 3 \\ 0 \\ -3 \end{bmatrix} \qquad x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$G = M^{-1}N \overset{\text{Jacobi}}{=} D^{-1}\boxed{(D-A)}$$

$$x^{(k)} = G x^{(k-1)} + M^{-1} b.$$

$$g_{ij}$$

$$D-A = \begin{bmatrix} 0 & -\alpha_{12} & -\alpha_{13} & \cdots & -\alpha_{16} \\ -\alpha_{21} & 0 & -\alpha_{23} & \cdots & -\alpha_{26} \\ & & \ddots & & \\ -\alpha_{61} & -\alpha_{62} & \cdots & -\alpha_{65} & 0 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} \frac{1}{\alpha_{11}} & & & 0 \\ & \frac{1}{\alpha_{22}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\alpha_{66}} \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & -\frac{\alpha_{12}}{\alpha_{11}} & -\frac{\alpha_{13}}{\alpha_{11}} & \cdots & -\frac{\alpha_{16}}{\alpha_{11}} \\ -\frac{\alpha_{21}}{\alpha_{22}} & 0 & -\frac{\alpha_{23}}{\alpha_{22}} & \cdots & -\frac{\alpha_{26}}{\alpha_{22}} \\ \vdots & & \vdots & & \vdots \\ -\frac{\alpha_{61}}{\alpha_{66}} & -\frac{\alpha_{62}}{\alpha_{66}} & \cdots & -\frac{\alpha_{65}}{\alpha_{66}} & 0 \end{bmatrix}$$

$$x^{(k-1)} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ \vdots \\ x_6^{(k-1)} \end{bmatrix}$$

$$\underset{(6\times 6)\times(6\times 1)}{\left[G x^{(k-1)}\right]_i} = -\sum_{\substack{j=1\\ i\neq j}}^{6} \frac{\alpha_{ij}}{\alpha_{ii}} x_j^{(k-1)} \quad , i=1,\dots,6$$

$$\left[M^{-1} b\right]_i = \left[D^{-1} b\right]_i = \begin{bmatrix} \frac{1}{\alpha_{11}} & & 0 \\ & \ddots & \\ & & \frac{1}{\alpha_{66}} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{bmatrix} = \frac{b_i}{\alpha_{ii}}$$

Μέθοδος Jacobi σε συνιστώσες.

$$x_i^{(k)} = \frac{b_i}{\alpha_{ii}} - \sum_{\substack{j=1\\ i\neq j}}^{6} \frac{\alpha_{ij}}{\alpha_{ii}} x_j^{(k-1)} = \frac{1}{\alpha_{ii}}\Big(b_i \overbrace{- \sum_{\substack{j=1\\ i\neq j}}^{6} \alpha_{ij} x_j^{(k-1)}}^{S}\Big) \qquad i=1,\dots,6.$$

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_6^{(0)} \end{bmatrix}$$

$$\|X^{(k)} - X^{(k-1)}\|_2 = \sqrt{(x_1^{(k)} - x_1^{(k-1)})^2 + \cdots + (x_6^{(k)} - x_6^{(k-1)})^2}$$

x = [0, 0, 0, 0, 0, 0]

A = [[4, 1, 0, 0, 0, 1], [1, 4, 1, 0, 0, 0], ..., [1, 0, 0, 0, 1, 4]]

b = [3, 0, -3, 3, 0, -3]

TOL = 0.001

```
while ||Xk - x|| > TOL:
 for i in range(6):        <- i = 0, 1, 2, 3, 4, 5

    for j in range(6):
        s = 0
        if j != i:
            s = s - A[i,j] X[j]
    Xk[i] = (1/A[i,i]) * (b[i] + s)
```

$\neq$

$x = xk.$

2η εργαστηριακή

$f(x) = x^3 - 2x - 5$, $x^{(0)} = 5$ $\qquad f' = 3x^2 - 2$

$$x^{(k)} = x^{(k-1)} - \frac{f(x^{(k-1)})}{f'(x^{(k-1)})}$$

μέχρι $|x^{(k)} - x^{(k-1)}| < TOL = 10^{-6}$ ή $KMAX = 100$

Επιστροφή στις Splines.

Κυβικές Splines. $S_3(\Delta)$ $\quad \Delta=\{\overset{\alpha}{\overset{\shortparallel}{x_0}},x_1,\dots,\overset{b}{\overset{\shortparallel}{x_n}}\}$

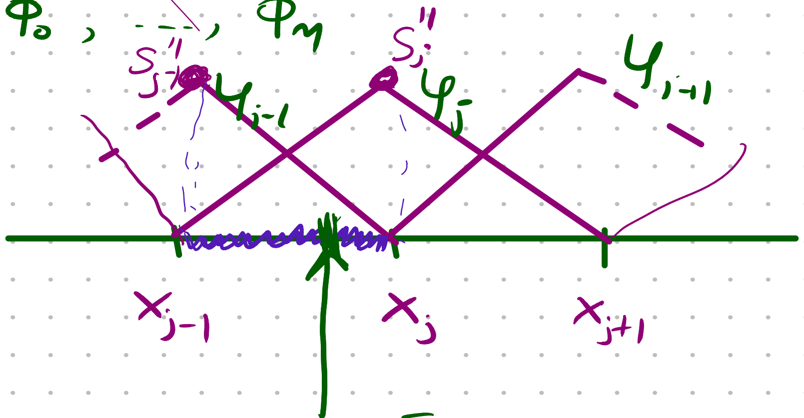
Έστω $s\in S_3(\Delta) \Rightarrow s''\in S_1(\Delta)$, $s\in C[\alpha,b]$ και $s''\big|_{[x_{j-1},x_j]}\in \mathbb{P}_1$

Άρα σε κάθε διάστημα $[x_{j-1},x_j]$

$$s''(x) = s''_{j-1}\boxed{\frac{x_j-x}{x_j-x_{j-1}}} + s''_j\boxed{\frac{x-x_{j-1}}{x_j-x_{j-1}}} \quad (*)$$

Βάση του $S_1(\Delta)$

$\phi_0,\dots,\phi_n$



$x\in[x_{j-1},x_j]$

Έστω $f\in S_1(\Delta)$ και $f\big|_{[x_{j-1},x_j]}$

$$f(x)=y_{j-1}\varphi_{j-1}(x)+y_j\varphi_j(x)$$

δεξί κλάδος της φ_{j-1}

Αριστερός κλάδος της φ_j

Ολοκληρώνουμε την (*) 2 φορές και παίρνουμε ομοιόμορφη διαμέριση πλάτους h.

$$S(x) = \frac{S''_{j-1}}{6h}(x_j - x)^3 + \frac{S''_j}{6h}(x - x_{j-1})^3 + \beta_{j1}x + \beta_{j0}$$

Αν απαιτήσουμε $S(x_j) = y_j$ $S(x_{j-1}) = y_{j-1}$ τότε θα έχουμε 2 εξισώσεις με 2 αγνώστους (β_{j1}, β_{j0})

$$\begin{cases} \beta_{j1}x_j + \beta_{j0} = y_j - \frac{S''_j}{6}h^2 \\ \beta_{j1}x_{j-1} + \beta_{j0} = y_{j-1} - \frac{S''_{j-1}}{6}h^2 \end{cases} \longrightarrow (\beta_{j1}, \beta_{j0})$$

$$\Rightarrow S(x) = \frac{S''_{j-1}}{6h}(x_j - x)^3 + \frac{S''_j}{6h}(x - x_{j-1})^3 + \left(\frac{y_j}{h} - S''_j\frac{h^2}{6}\right)(x - x_{j-1}) +$$

$$+ \left(\frac{y_{j-1}}{h} - S''_{j-1}\frac{h^2}{6}\right)(x_j - x) \quad \forall x \in [x_{j-1}, x_j]$$

Τέλος από $S'(x_j^-) = S'(x_j^+)$ (συνέχεια $1^{ης}$ παραγώγου)

$$S''_{j-1} + 4S''_j + S''_{j+1} = 6\,\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} \qquad j = 1, \dots, n-1$$

και επιπλέον $S_0'' = 0 = S_n''$

$$\begin{bmatrix} 1 & 0 & \cdots & & & & 0 \\ 1 & 4 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & \cdots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} S_0'' \\ S_1'' \\ \vdots \\ S_{n-1}'' \\ S_n'' \end{bmatrix} = \begin{bmatrix} 0 \\ 6\,\frac{y_2 - 2y_1 + y_0}{h^2} \\ 6\,\frac{y_3 - 2y_2 + y_1}{h^2} \\ \vdots \\ 6\,\frac{y_n - 2y_{n-1} + y_{n-2}}{h^2} \\ 0 \end{bmatrix}$$

$\exists$ μοναδική λύση για τις τιμές των 2ων παραγώγων.

<u>Θεώρημα</u>: Έστω $f \in C^4([a,b])$

και S η κυβική spline σε μια ομοιόμορφη διαμέριση Δ.

Τότε

$$\max_{a \le x \le b} |f^{(k)}(x) - S^{(k)}(x)| \le C_k h^{4-k} \max_{a \le x \le b} |f^{(4)}(x)| , \quad k = 0, \ldots, 3, \quad C_k \text{ σταθερές}$$

Για $k=0$

$$\max_{a \le x \le b} |f(x) - S(x)| \le C_0 h^4 \max_{a \le x \le b} |f^{(4)}(x)|.$$