ΜΕΜ-Θ602: Μαθηματική Χρηματοοικονομία

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

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Έστω $X_1, X_2, \ldots$ ανεξάρτητες και ισόνομες τυχαίες μεταβλητές στο φιλτραρισμένο χώρο πιθανότητας

$$(\Omega, \mathcal{F} = 2^{\Omega}, \{\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)\}, \mathbb{P})$$

και

$$\mathbb{P}[X_1 = -1] = 1/2 = \mathbb{P}[X_1 = 1].$$

Ορίζουμε την στοχαστική διαδικασία $\{S_t\}_{t \in \mathbb{T} = \{0,1,2,\ldots\}}$ ως

$$S_t = \begin{cases} 0, & t = 0, \\ \sum_{t=1}^{n} X_t, & t = 1, 2, \ldots \end{cases}$$

$S_{t+1} = S_t + X_{t+1}$

Εξετάστε εάν η $M_t = S_t^3 - 3tS_t, t \in \mathbb{T}$ είναι martingale.

$\mathbb{E}[M_{t+1} | \mathcal{F}_t] = M_t$?

$$M_{t+1} = S_{t+1}^3 - 3(t+1)S_{t+1} = (S_t + X_{t+1})^3 - 3(t+1)(S_t + X_{t+1}) =$$

$$= S_t^3 + 3S_t^2 X_{t+1} + 3S_t X_{t+1}^2 + X_{t+1}^3 - 3(t+1)X_{t+1} - 3tS_t - 3S_t =$$

$$= M_t + \underset{①}{3S_t^2 X_{t+1}} + \underset{②}{3S_t X_{t+1}^2} + \underset{③}{X_{t+1}^3} - \underset{④}{3(t+1)X_{t+1}} - \underset{⑤}{3S_t}$$

ανεξ.

① $\mathbb{E}[S_t^2 X_{t+1} \mid \mathcal{F}_t] = S_t^2 \mathbb{E}[X_{t+1} \mid \mathcal{F}_t] = S_t^2 \mathbb{E}[X_{t+1}]$

$\mathbb{E}[X_{t+1}] = p(X_{t+1} = 1) \cdot 1 + p(X_{t+1} = -1) \cdot (-1) = 0$

② $\mathbb{E}[S_t X_{t+1}^2 \mid \mathcal{F}_t] = S_t \mathbb{E}[X_{t+1}^2] = S_t \big(p(X_{t+1} = 1) \cdot 1^2 +$

$+ p(X_{t+1} = -1) \cdot (-1)^2 \big) = S_t$

③ $\mathbb{E}[X_{t+1}^3 | \mathcal{F}_t] = \mathbb{E}[X_{t+1}^3] = \frac{1}{2}\cdot(-1)^3 + \frac{1}{2}1^3 = 0$

④ $\mathbb{E} = 0$

⑤ $\mathbb{E}[S_t | \mathcal{F}_t] = S_t$

Άρα $\mathbb{E}[M_{t+1} | \mathcal{F}_t] = \mathbb{E}\cdot[M_t | \mathcal{F}_t] + 0 = M_t$

άρα M_t martingale.

Έστω η ακόλουθη στοχαστική διαδικασία

$$Y_t = e^{W_t},\ t \in \mathbb{T} = [0,1]$$

Υπολογίστε την συνδιασπορά των Y_t, Y_s, για $t, s \in (0,1)$

$Cov(W_t, W_s) = \min\{t, s\}$

$W_t \sim N(0,t)$

$X \sim N(\mu, \sigma^2)$

$\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}t^2\sigma^2}$

$\mathbb{E}[Y_t] = \mathbb{E}[e^{W_t}] = e^{\frac{t}{2}}$

$t > s$

$Cov[Y_t, Y_s] = \mathbb{E}[Y_t Y_s] - \mathbb{E}[Y_t]\mathbb{E}[Y_s]$

$\mathbb{E}[Y_t Y_s] = \mathbb{E}[e^{W_t} e^{W_s}] = \mathbb{E}[e^{W_t - W_s} e^{2W_s}] =$

$= \mathbb{E}[e^{W_t - W_s}]\mathbb{E}[e^{2W_s}] = *$

$W_t - W_s \sim N(0, t-s)$

$$\circledast = e^{\frac{1}{2}(t-s)} e^{\frac{1}{2}4s} = e^{1/2\, t + 3/2\, s}$$

Άρα $\quad Cov[Y_t, Y_s] = e^{\frac{1}{2}(t+3s)} - e^{\frac{1}{2}(t+s)}, \quad t \geq s$

$$Corr[Y_t, Y_s] = \frac{Cov[Y_t, Y_s]}{std(Y_t)\, std(Y_s)} = \frac{e^{\frac{1}{2}(t+3s)} - e^{1/2(t+s)}}{e^{t+s} - e^{1/2(t+s)}}$$

$$Var[Y_t] = \mathbb{E}\left[(Y_t - \mathbb{E}[Y_t])^2\right] =$$

$$= \mathbb{E}\left[(e^{W_t} - e^{t/2})^2\right]$$

Έστω η ακόλουθη στοχαστική διαδικασία

$$Y_t = e^{tW_t},\ t \in \mathbb{T} = [0,1]$$

Υπολογίστε την συνδιασπορά των Y_t, Y_s, για $t, s \in (0,1)$

$t \geqslant s$

$$\mathbb{E}[Y_t] = \mathbb{E}[e^{tW_t}] = e^{\frac{1}{2}t^2 t} = e^{\frac{1}{2}t^3} \neq 0$$

$$\mathbb{E}[Y_t Y_s] = \mathbb{E}[e^{tW_t} e^{sW_s}] =$$

$$= \mathbb{E}[e^{t(W_t - W_s)} e^{(t+s)W_s}] =$$

$$= \mathbb{E}[e^{t(\overbrace{W_t - W_s}^{\sim N(0,\, t-s)})}] \, \mathbb{E}[e^{(t+s)\overbrace{W_s}^{\sim N(0,\, s)}}]$$

$$= e^{\frac{1}{2}t^2(t-s)} e^{\frac{1}{2}(t+s)^2 \cdot s}$$

$$\mathrm{Cov}[Y_t, Y_s] = \mathbb{E}[Y_t Y_s] - \mathbb{E}[Y_t]\,\mathbb{E}[Y_s]$$

Υπολογίστε την διασπορά του ακόλουθου ολοκληρώματος κατά Ito

$$I = \int_0^{2\pi} \sqrt{|W_t|}\, dW_t$$

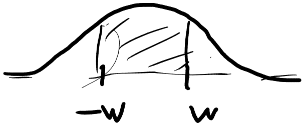
$$\mathbb{E}[I] = 0$$

$$\mathrm{Var}[I] = \mathbb{E}[I^2] = \mathbb{E}[|I|^2] = \|I\|_{L^2(\Omega)}^2 \overset{\text{isometry}}{=}$$

$$= \left\|\sqrt{|W_t|}\right\|_{M^2(\Omega)} = \mathbb{E}\left[\int_0^{2\pi} \left(\sqrt{|W_t|}\right)^2 dt\right] =$$

$$= \mathbb{E}\left[\int_0^{2\pi} |W_t|\, dt\right] = \int_0^{2\pi} \mathbb{E}\left[|W_t|\right] dt$$

$$\mathbb{P}\left[\,|W_t| \le w\,\right] = F_{|W_t|}(w) =$$



$$= \mathbb{P}\left[-w \le W_t \le w\right] =$$

$$= \mathbb{P}\left[W_t \le w\right] - \mathbb{P}\left[W_t \le -w\right] = 2\,\mathbb{P}\left[W_t \le w\right] - 1$$

$$p_{|W_t|}(w) = 2\,p_{W_1}(w) = 2\,\frac{1}{\sqrt{2\pi t}}\,e^{-\frac{1}{2}\left(\frac{w}{\sqrt{t}}\right)^2} \qquad F''_{W_t}(w)$$

$$E[|W_t|] = \int_0^\infty w\, 2\, \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\left(\frac{w}{\sqrt{t}}\right)^2} dw =$$

$$y = \frac{w}{\sqrt{t}} \qquad dw = \sqrt{t}\, dy$$

$$= 2\sqrt{\frac{t}{2\pi}} \int_0^\infty y\, e^{-\frac{1}{2}y^2} dy =$$

$$= -2\sqrt{\frac{t}{2\pi}} \int_0^\infty \left(e^{-\frac{1}{2}y^2}\right)' dy = 2\sqrt{\frac{t}{2\pi}}$$

$$Var[I] = 2\sqrt{\frac{1}{2\pi}} \int_0^{2\pi} \sqrt{t}\, dt$$

Δείξτε ότι η ακόλουθη στοχαστική διαδικασία B_t

$$B_t = \int_0^{\sqrt{t}} \sqrt{2s} dW_s$$

είναι κίνηση Brown.

$B_0 = 0$

$B_{t_2} - B_{t_1}$, $B_{s_2} - B_{s_1}$ ανεξ. αν $[t_1, t_2] \cap [s_1, s_2] = \emptyset$

Θέλουμε ν.δ.ο $B_t \sim N(0, t)$

$\int$ $[0, \sqrt{t}]$ $T = \sqrt{t}$

$$I_n = \sum_{k=0}^{n-1} \sqrt{2s_k} \underbrace{\left(W_{s_{k+1}} - W_{s_k}\right)}$$

$s_0, s_1, s_2, \ldots, s_n$ $\delta s = \frac{\sqrt{t}}{n}$

$s_k = k\delta s = k\frac{\sqrt{t}}{n}$, $k = 0, \ldots, n$

$$W_{s_{k+1}} - W_{s_k} \sim \mathcal{N}(0, \delta s)$$

$$\sum_{k=0}^{n-1} \sqrt{2 s_k}\,\left(W_{s_{k+1}} - W_{s_k}\right) \sim \mathcal{N}\left(0, \sum_{k=0}^{n-1} 2 s_k \cdot \delta s\right)$$

$$\sigma^2 = 2\delta s \sum_{k=0}^{n-1} s_k = 2(\delta s)^2 \sum_{k=0}^{n-1} k = \cancel{2}(\delta s)^2 \frac{1}{\cancel{2}}(n-1)\cdot n =$$

$$s_k = k\delta s$$

$$= n\delta s \cdot (n-1)\delta s =$$

$$= \sqrt{t}\left(\sqrt{t} - \frac{\sqrt{t}}{n}\right) \xrightarrow{n\to\infty} t$$