ΜΕΜ-Θ602: Μαθηματική Χρηματοοικονομία

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

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$S_t^* = e^{-rt} S_t$

Στο φυσικό μέτρο πιθανότητας έχουμε

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad \Leftrightarrow \quad \frac{dS_t^*}{S_t^*} = (\mu - r)dt + \sigma dW_t$$

Στο ισοδύναμο μέτρο martingale έχουμε

$\mathbb{Q} \sim \mathbb{P}$, S_t^* $\mathbb{Q}$-martingale

$$\frac{dS_t^*}{S_t^*} = \sigma dW_t^{\mathbb{Q}}$$

Θα δείξουμε ότι

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}}$$

$dS_t^* =$

$$dS_t^* = d(e^{-rt}S_t) = d(e^{-rt})S_t + e^{-rt}dS_t + d[e^{-rt}, S_t]$$

$$= -re^{-rt}S_t\,dt + e^{-rt}dS_t + 0$$

$$\frac{dS_t^*}{S_t^*} = \boxed{\frac{-r\cancel{e^{-rt}}S_t\,dt + \cancel{e^{-rt}}dS_t}{\cancel{e^{-rt}}S_t} = \sigma\, dW_t^Q}$$

$$\frac{dS_t}{S_t} = r\,dt + \sigma\, dW_t^Q$$

$$d[e^{-rt}, S_t] = de^{-rt}\,dS_t = -re^{-rt}dt\,(\ldots dt + \ldots dW_t^Q) = 0$$

Θεωρούμε την συνάρτηση $V(S,t),\ S \geq 0,\ t \in [0,T]$ $\quad V_t = V(S_t, t)$

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

$$dS_t = \underbrace{\overset{= r}{\mu} S_t}_{rS_t} dt + \overbrace{\sigma S_t}\, \underbrace{dW_t}_{dW_t^Q}$$

$$V(S,T) = (S-K)_+ \quad \text{terminal condition.}$$

$$dV(S_t,t) = \left\{ \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right\} dt + \sigma S_t \frac{\partial V}{\partial S} dW_t^Q \quad \Big\}\ \text{Ito's Lemma.}$$

$$V^* = e^{-rt} V$$

$$V^* = V^*(t, V(t))$$

$$dV^* = \frac{\partial V^*}{\partial t} dt + \frac{\partial V^*}{\partial V} dV + \frac{1}{2} \frac{\partial^2 V^*}{\partial V^2} (dV)^2 + \frac{\partial^2 V^*}{\partial t \partial V} \underbrace{dt\, dV}_{=0}$$

$$\frac{\partial V^*}{\partial t} = -re^{-rt} V \,, \quad \frac{\partial V^*}{\partial V} = e^{-rt} \,, \quad \frac{\partial^2 V^*}{\partial V^2} = 0$$

$$\boxed{dV^* = -re^{-rt} V dt + e^{-rt} dV}$$

$$dV^* = e^{-rt} \left\{ \frac{\partial V}{\partial t} + r S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV \right\} dt$$

$$+ e^{-rt} \sigma S_t \frac{\partial V}{\partial S} dW_t^Q$$

Εστω V είναι λύση της εξισωσης B-S

$$dV^* = e^{-rt}\sigma S_t \frac{\partial V}{\partial S}\, dW_t^Q$$

$$\int_0^T dV^* = \int_0^T e^{-rt}\sigma S_t \frac{\partial V}{\partial S}\, dW_t^Q = I$$

$$V_0^* = V_0$$

$$V_0^* = e^{-r\cdot 0} V_0$$

$$V_T^* - V_0^* = I \Rightarrow V_T^* = V_0 + I$$

$$\mathbb{E}_Q[V_T^* | S_0] = V_0 + \overbrace{\mathbb{E}_Q[I / S_0]}^{=0}$$

$$\mathbb{E}_Q[V_T^* | S_0] = V_0$$

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (*) \quad , S \geq 0 \, , \; t \in [0,T]$$

$$V(S,T) = (S-K)_+$$

$T - t$

Αλλαγή μεταβλητής $t = T - \frac{2\tau}{\sigma^2}$ ή $\tau = \frac{\sigma^2}{2}(T-t)$

όταν $t = 0 \Rightarrow \tau = T\frac{\sigma^2}{2}$

$t = T \Rightarrow \tau = 0.$

Αλλαγή μεταβλητής $S = e^{\overset{x \in \mathbb{R}}{x}}$ ή $x = \ln S$

$S > 0$

$$V(S,t) = V(S(x), t(\tau)) = u(x,\tau) \quad , \quad x \in \mathbb{R} \, , \; \tau \in \left[0, T\frac{\sigma^2}{2}\right]$$

$$\frac{\partial V}{\partial t} = \frac{\partial u}{\partial \tau}\frac{d\tau}{dt} = -\frac{\sigma^2}{2}\frac{\partial u}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = \frac{\partial u}{\partial x}\frac{dx}{dS} = \frac{1}{S}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial u}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial u}{\partial x} + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial u}{\partial x}\right) =$$

$$= -\frac{1}{S^2}\frac{\partial u}{\partial x} + \frac{1}{S}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{dx}{dS} = -\frac{1}{S^2}\frac{\partial u}{\partial x} + \frac{1}{S^2}\frac{\partial^2 u}{\partial x^2}$$

Αντικαθ. στη (*)

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial u}{\partial x} - \frac{2r}{\sigma^2}u \quad , \quad u(x,0) = (e^x - K)_+$$

$$k = \frac{2r}{\sigma^2}$$

strike price

(**)

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k-1)\frac{\partial u}{\partial x} - ku, \qquad u(x,0) = (e^x - K)_+$$

Αλλαγή μεταβλητής

$$u(x,\tau) = e^{\alpha x + \beta \tau} w(x,\tau) = \phi(x,\tau)\, w(x,\tau)$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial \phi}{\partial \tau} w + \phi \frac{\partial w}{\partial \tau} = \beta \phi w + \phi \frac{\partial w}{\partial \tau}$$

$$\frac{\partial u}{\partial x} = \frac{\partial \phi}{\partial x} w + \phi \frac{\partial w}{\partial x} = \alpha \phi w + \phi \frac{\partial w}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \alpha^2 \phi w + 2\alpha\phi \frac{\partial w}{\partial x} + \phi \frac{\partial^2 w}{\partial x^2}$$

$u(x,0)$

(**) $\Rightarrow$

$$\beta\phi w + \phi \frac{\partial w}{\partial \tau} = \alpha^2 \phi w + 2\alpha\phi \frac{\partial w}{\partial x} + \phi \frac{\partial^2 w}{\partial x^2}$$

$$+ (k-1)\left(\alpha\phi w + \phi \frac{\partial w}{\partial x}\right) - k\phi w$$

Εξίσωση θερμότητας

$$\left.\begin{aligned} \alpha &= -\frac{1}{2}(k-1) \\ \beta &= -\frac{1}{4}(k+1)^2 \end{aligned}\right\} \Rightarrow \left[\begin{aligned} &\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, \quad x \in \mathbb{R},\ t \in \left[0, \frac{\sigma^2}{2} T\right] \\ &w(x,0) = e^{-\alpha x} u(x,0) \end{aligned}\right.$$

$$w \to u = e^{\alpha x + \beta \tau} w \longrightarrow V$$

$$S = e^x$$

$$t = T - \frac{2\tau}{\sigma^2}$$