A deep implicit-explicit minimizing movement method for option pricing in jump-diffusion models

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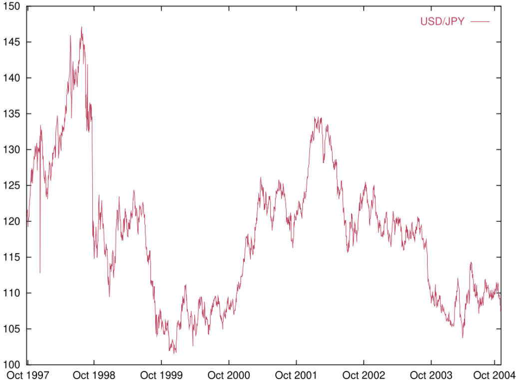
Costas Smaragdakis

Joint work with **Antonis Papapantoleon** & **Manolis Georgoulis**

UNIVERSITY OF THE AEGEAN
SCHOOL OF SCIENCES
DEPARTMENT OF STATISTICS AND ACTUARIAL–FINANCIAL MATHEMATICS

- Pricing options in scenarios where the underlying stock values exhibit discontinuities.
- Our focus is on options involving a large number of underlying assets.



We assume d correlated stocks with values modelled by the stochastic processes

$$S_t^{(i)} = S_0^{(i)} \exp\left(\mu_i t + \sigma_i W_t^{(i)} + \sum_{k=1}^{N_t} Z_k^{(i)}\right),\ t \in \mathbb{T} = [0, T],\ i = 1, \ldots, d$$

$$W_t^{(i)} : \text{Brownian motions}$$

$$Z_k^{(i)} : \text{Normal random variables}$$

$$N_t : \text{Poisson process}$$

$$\text{Corr}[W_t^{(i)}, W_t^{(j)}] = \rho_{ij} \in [-1, 1], \quad \text{Corr}[Z_k^{(i)}, Z_k^{(j)}] = \rho_{Jij} \in [-1, 1]$$

Payoff function

- $\{\alpha_i\}_{i=1,\ldots,d}$: weights on underlyings ($\sum_i \alpha_i = 1$)
- K : the strike price

$$\text{Payoff}(S) = \left(\sum_{i=1}^{d} \alpha_i S_i - K\right)^+$$

Moneynesses : $x_i = S_i/K$

$$\text{Payoff}(x) = \left(\sum_{i=1}^{d} \alpha_i x_i - 1\right)^+$$

Arbitrage-free price

$\mathbb{E}^{\mathbb{Q}}[e^{-rt}\text{Payoff}(S_t)|S_0]$, where t expresses the time of maturity.

Partial Integro-differential Equation

Using FTAP and Feynman-Kac the option price is provided by:

$$\partial_t u(t,x) + \mathcal{A}u(t,x) = 0,\ t > 0,\ x \in [0,\infty)^d$$

$$u(0,x) = u_0(x) = \text{Payoff}(x),\ x \in [0,\infty)^d$$

- $\mathcal{A}$: PIDE operator

$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru - I_\nu[u]$$

where $a_{ij}(x) = a_{ji}(x),\ i,j = 1,\ldots,d$ and $I_\nu[u] = \int_{\mathbb{R}^d}[u(t,xe^z) - u(t,x)]\nu(dz)$.

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$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru - I_\nu[u],\ \ I_\nu[u] = \int_{\mathbb{R}^d} [u(t, xe^z) - u(t,x)]\nu(dz)$$

We rewrite the operator as follows

$$\mathcal{A}u = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d}\left(b_i(x) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u}{\partial x_i} + ru - I_\nu[u]$$

$$\mathcal{A}u = \mathcal{L}u + f[u]$$

$$\begin{aligned} \mathcal{L}u &= -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + ru \quad \text{(symmetric)} \\ f[u] &= \sum_{i=1}^{d}\left(b_i(x) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u}{\partial x_i} - I_\nu[u] \quad \text{(remainder)} \end{aligned}$$

Suppose we would like to estimate the solution of the following equation:

$$\begin{aligned} \partial_t u(t,x) + \mathcal{L}u(t,x) + f[u(t,x)] &= 0, \quad x \in [0, x_{\max}]^d = \Omega, \quad t \in [0,T] \\ u(0,x) &= \text{Payoff}(x) \end{aligned}$$

We consider a time subdivision of the time interval $[0,T]$

$$\tau = T/n, \quad t_k = k\tau, \quad k = 0, \dots, n$$

$$u^k(x) \doteq u(t_k, x)$$

Implicit–Explicit BDF-p

$$\frac{\beta_p u^k - \sum_{j=0}^{p-1} \beta_j u^{k-j-1}}{\tau} + \mathcal{L}u^k + \sum_{j=0}^{p-1} \gamma_j f[u^{k-j-1}] = 0$$

Implicit–Explicit BDF-p

- Approximate $u^1, u^2, \ldots, u^p$ using the implicit-explicit Euler method

$$\beta_p u - \sum_{j=0}^{p-1} \beta_j u^{k-j-1} + \tau \mathcal{L} u + \tau \sum_{j=0}^{p-1} \gamma_j f[u^{k-j-1}] = 0, \quad k \geq p$$

Minimization Problem : $u^{k-p}, \ldots, u^{k-1} \to u^k$

$$L = \frac{1}{2}\left(\beta_p u - \sum_{j=0}^{p-1} \beta_j u^{k-j-1}\right)^2 + \tau \mathcal{E}[u] + \tau \sum_{j=0}^{p-1} \gamma_j f[u^{k-j-1}] u$$

Dirichlet energy functional : $\mathcal{E}[u] = \frac{1}{2}\left(\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + r u^2\right)$

$$\mathcal{C}[u] \doteq \frac{1}{2}\left\|\beta_p u - \sum_{j=0}^{p-1} \beta_j u^{k-j-1}\right\|^2_{L^2(\Omega)} + \tau \int_\Omega \mathcal{E}[u] dx + \tau \int_\Omega \sum_{j=0}^{p-1} \gamma_j f[u^{k-j-1}] u \, dx \to \min$$

ANN Representation

We approximate the solution at the step t_k by a ANN with parameters θ^k.

$$u^k(x) \approx U^k(x;\theta^k)$$

H. Georgoulis, M. Loulakis, and A. Tsiourvas (2023).

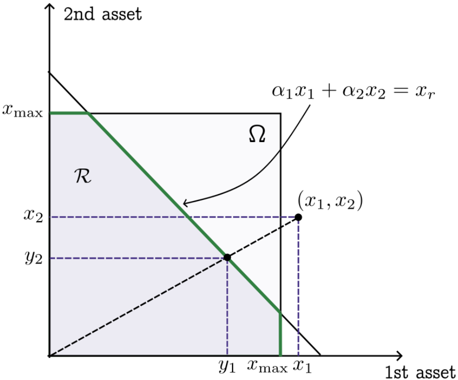
Discretized Cost Functional

$$\begin{aligned}\mathscr{C}_k(\theta) \quad := \quad & \frac{(x_{\max})^d}{N}\sum_{i=1}^{N}\Bigg\{\frac{1}{2}\Bigg[\beta_p U^k(x^i;\theta) - \sum_{j=0}^{p-1}\beta_j U^{j(k)}(x^i;\theta^{j(k)})\Bigg]^2 \\ + \quad & \tau\mathcal{E}[U^k(x^i;\theta)] + \tau\sum_{j=0}^{p-1}\gamma_j f[U^{j(k)}(x^i;\theta^{j(k)})]U^{j(k)}(x^i;\theta)\Bigg\}\end{aligned}$$

where $j(k) = k - j - 1$

<u>Optimization step</u> : $\theta^k \leftarrow \min_\theta \mathscr{C}_k(\theta)$

$$y := q(x)x, \quad q(x) = \begin{cases} x_{\max}/\max\{x_i\}, & \text{if } \max\{x_i\} \geq \max\left(\sum_{i=1}^d \alpha_i x_i, x_r\right) x_{\max}/x_r \\ x_r/\max\left(\sum_{i=1}^d \alpha_i x_i, x_r\right), & \text{otherwise.} \end{cases}$$



$$x \in \mathbb{R}_+^d \to y \in \mathcal{R} \cup \partial\mathcal{R} \to U^k(y;\theta^k) \to U^k(x;\theta^k)$$

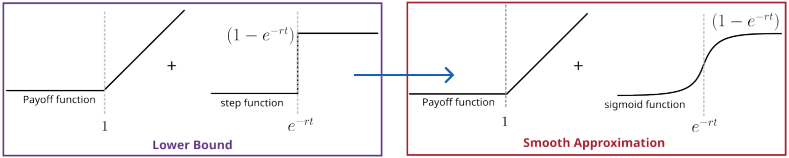
$$U^k(x;\theta^k) \approx U^k(y;\theta^k) + \sum_{i=1}^d \alpha_i (x_i - y_i)$$

(Approximation of the solution at time t_k) = (Lower Bound at time t_k) + (positive function)

$$U^k(y;\theta^k) = \tilde{v}(t_k, y) + w^k(y;\theta^k), \quad y \in \mathcal{R} \cup \partial\mathcal{R}$$

Lower bound $v(t,x)$

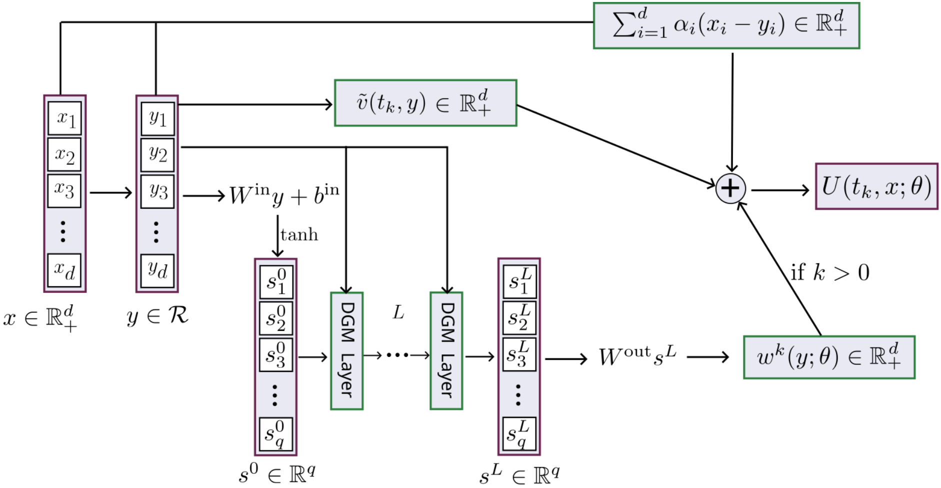
$$v(t,x) = \text{Payoff}(x) + (1 - e^{-rt})H\left(\sum_{i=1}^{d} \alpha_i x_i - e^{-rt}\right)$$



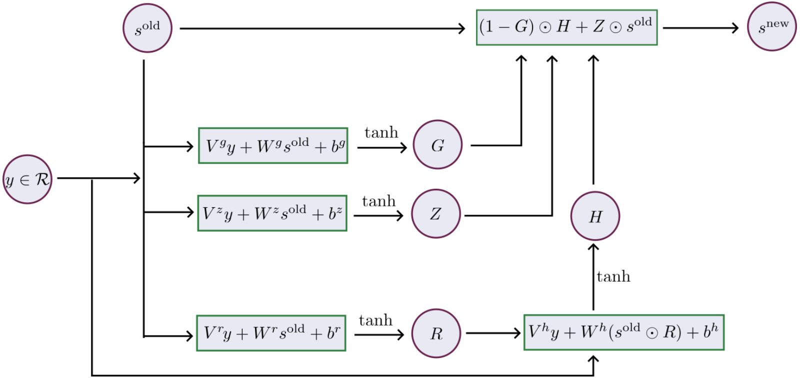
Smooth approximation $\tilde{v}(t,x;\eta)$

$$\tilde{v}(t,x;\eta) = \text{Payoff}(x) + (1 - e^{-rt})\text{Sig}\left(\sum_{i=1}^{d} \alpha_i x_i - e^{-rt};\eta\right),\ \text{Sig}(x;\eta) = (1 + e^{-\eta x})^{-1},\ \eta > 0$$

$$\lim_{\eta\to\infty} \tilde{v}(t,x;\eta) = v(t,x)$$


$\sum_{i=1}^{d} \alpha_i(x_i - y_i) \in \mathbb{R}_+^d$
$\tilde{v}(t_k, y) \in \mathbb{R}_+^d$
$U(t_k, x; \theta)$
x_1
x_2
x_3
x_d
y_1
y_2
y_3
y_d
$W^{\text{in}} y + b^{\text{in}}$
tanh
$x \in \mathbb{R}_+^d$
$y \in \mathcal{R}$
s_1^0
s_2^0
s_3^0
s_q^0
DGM Layer
L
DGM Layer
s_1^L
s_2^L
s_3^L
s_q^L
$W^{\text{out}} s^L$
$w^k(y; \theta) \in \mathbb{R}_+^d$
if $k > 0$
$s^0 \in \mathbb{R}^q$
$s^L \in \mathbb{R}^q$

DGM Layer


s^{old}
$(1 - G) \odot H + Z \odot s^{\text{old}}$
s^{new}
$y \in \mathcal{R}$
$V^g y + W^g s^{\text{old}} + b^g$
tanh
G
$V^z y + W^z s^{\text{old}} + b^z$
tanh
Z
H
tanh
$V^r y + W^r s^{\text{old}} + b^r$
tanh
R
$V^h y + W^h (s^{\text{old}} \odot R) + b^h$

$$\mathscr{C}_k(\theta) := \frac{(x_{\max})^d}{N}\sum_{i=1}^{N}\Bigg\{\frac{1}{2}\Big[\beta_p U^k(x^i;\theta) - \sum_{j=0}^{p-1}\beta_j U^{j(k)}(x^i;\theta^{j(k)})\Big]^2$$

$$+ \quad \tau\mathcal{E}[U^k(x^i;\theta)] + \tau\sum_{j=0}^{p-1}\gamma_j f[U^{j(k)}(x^i;\theta^{j(k)})]U^{j(k)}(x^i;\theta)\Bigg\}$$

$$\mathcal{E}[u] = \frac{1}{2}\left(\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + ru^2\right), \quad f[u] = \sum_{i=1}^{d}\left(b_i(x) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u}{\partial x_i} - I_\nu[u]$$

Merton model

$$a_{ij}(x) = \frac{1}{2}\sigma_i\rho_{ij}\sigma_j x_i x_j, \qquad b_i(x) = [-r + \frac{1}{2}\sigma_i^2 - \lambda\exp(\mu_{Ji} + \frac{1}{2}\sigma_{Ji}^2) - \lambda]x_i$$

$$I_\nu[u] = \lambda\int_{\mathbb{R}^d}(u(t,xe^z) - u(t,x))p(z)dz, \quad p(z) : \text{multivariate normal pdf}$$

$$\sigma_i = 0.5,\ \rho_{ij} = \delta_{ij} + 0.5(1-\delta_{ij}),\ \ i,j = 1,\dots,d,\ r = 0.05 \quad \text{(diffusion parameters)}$$

$$\lambda = 1,\ \mu_{Ji} = 0,\ \sigma_{Ji} = 0.5,\ \rho_{Jij} = \delta_{ij} + 0.2(1-\delta_{ij}),\ i,j = 1,\dots,d \quad \text{(jump parameters)}$$

$$\sum_{j=1}^{d} \gamma_j I_\nu[U^{j(k)}(x;\theta^{j(k)})], \quad I_\nu[U^{j(k)}(x;\theta^{j(k)})] = \lambda \int_{\mathbb{R}^d} (U^{j(k)}(xe^z;\theta^{j(k)}) - U^{j(k)}(x;\theta^{j(k)}))p(z)dz$$

Gauss-Hermite quadrature

In Merton model, the integral is simply the expected value of the function $h^{j(k)}(x,z) = U^{j(k)}(xe^z) - U^{j(k)}(x)$ multiplied by the Poisson parameter λ.

- Singular Value Decomposition (SVD) of Σ_J

$$\Sigma_J = U\Lambda U^T = U\Lambda^{1/2}\Lambda^{1/2}U^T = VV^T,$$

where $V = U\Lambda^{1/2}$.

- change of variable $Z - \mu = \sqrt{2}VY$

$$\begin{aligned} I_\nu[U^{j(k)}(x^i)] = \lambda \int_{\mathbb{R}^d} h^{j(k)}(x^i,z)p(z)dz &= \lambda\pi^{-d/2} \int_{\mathbb{R}^d} \exp(y^T y) h^{j(k)}(x^i, \mu + \sqrt{2}Vy)dy \\ &\approx \lambda\pi^{-d/2} \sum_{\mathbf{r}\in\Theta_p} h^{j(k)}(x^i, \mu + \sqrt{2}Vy^{\mathbf{r}})W^{\mathbf{r}}, \end{aligned}$$

$$\sum_{j=1}^{d} \gamma_j I_\nu[U^{j(k)}(x;\theta^{j(k)})], \quad I_\nu[U^{j(k)}(x;\theta^{j(k)})] = \lambda \int_{\mathbb{R}^d} (U^{j(k)}(xe^z;\theta^{j(k)}) - U^{j(k)}(x;\theta^{j(k)}))p(z)dz$$

Unbiased estimator of the integral operator

$$\min_{\phi\in\Phi} \mathbb{E}\left[\mathcal{I}^k(x;\phi) - \sum_{j=1}^{p-1} \gamma_j I_\nu[U^{j(k)}(x;\theta^{j(k)})]\right]^2$$

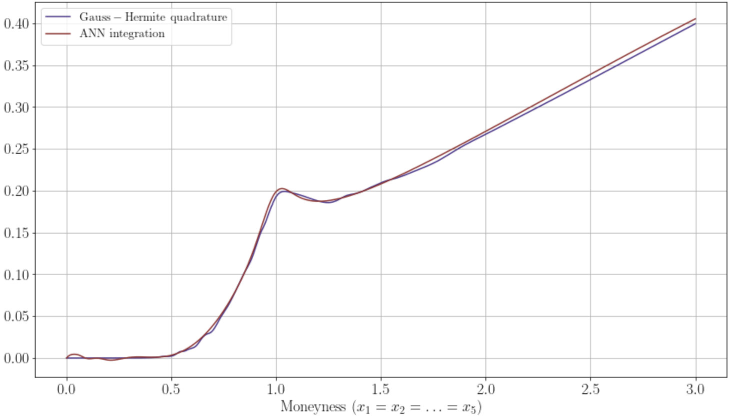
Additional term in the cost functional

Optimizer : (θ^k, ϕ^k)

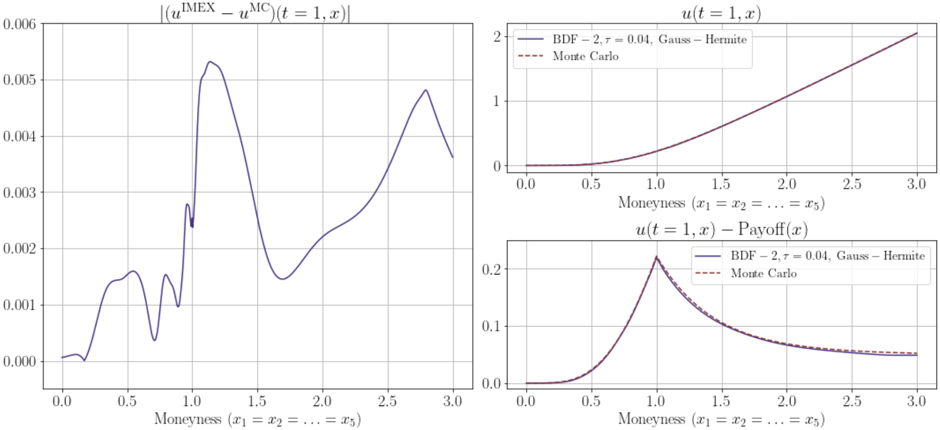
$$\frac{(x_{\max})^d}{N} \sum_{i=1}^{N} \left[\mathcal{I}^k(x^i;\phi) - \frac{\lambda}{M} \sum_{r=1}^{M} \sum_{j=1}^{p-1} \gamma_j h^{j(k)}(x^i, z^r)\right]^2$$

where $\{z^r\}_{r=1}^M$ are sampled from $p(z)$.

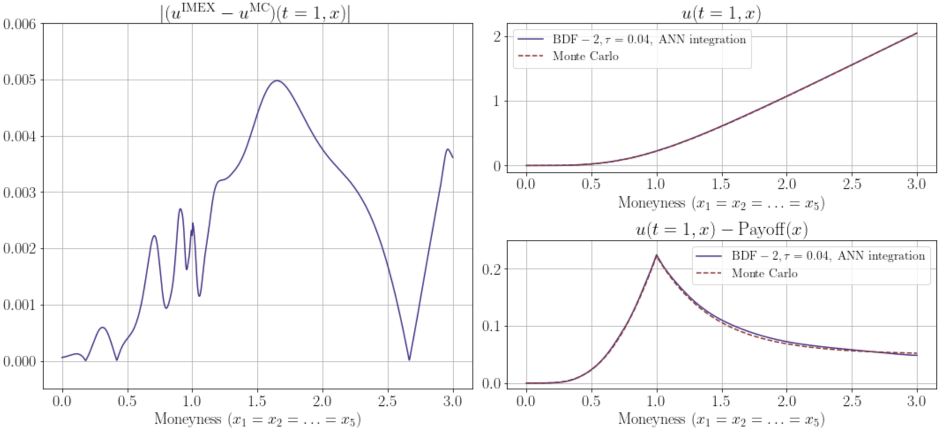
Gauss Hermite Quadrature vs ANN Integration



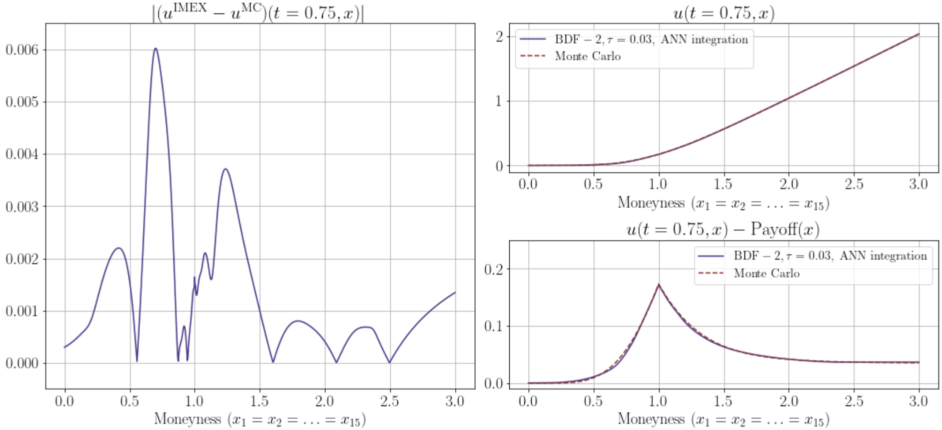
5 assets – 12 months – BDF-2 – Gauss Hermite Quadrature



5 assets – 12 months –BDF-2 – ANN Integration



15 assets – 9 months – BDF-2 – ANN Integration



- We proposed a new deep implicit–explicit minimizing movement method for option pricing.
- Our method is capable of accurately approximating the solutions of the partial integro-differential equations (PIDEs) that arise in European basket call options.
- To evaluate the effectiveness of our method, we compared its results with those obtained using Monte-Carlo simulations.

- We are working on the multivariate variance Gamma model (infinite jump activity) and introducing an improved truncation approach.

Acknowledgement

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Thank you for your attention

E.H. Georgoulis, M. Loulakis, and A. Tsiourvas: **Discrete gradient flow approximations of high dimensional evolution partial differential equations via deep neural networks.** Communications in Nonlinear Science and Numerical Simulation, 117:106893 (11), 2023.

E.H. Georgoulis, A. Papapantoleon, C. Smaragdakis: **A deep implicit-explicit minimizing movement method for option pricing in jump-diffusion models.** Preprint and submitted for publication, 2024 [arXiv:2401.06740].

On Thursday, at 10.00
Jasper Rou: **Deep gradient flow methods for option pricing in diffusion models.**