$$\alpha y'' + by' + cy = 0$$
$$y(t_0) = y_0$$
$$y'(t_0) = y_0'$$

$$y = e^{rt}$$

$$\alpha r^2 + br + c = 0 \quad \begin{cases} \Delta > 0 \\ \Delta = 0 \\ \Delta < 0 \end{cases}$$

Γενική λύση

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$
$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$$

$\Delta < 0 \quad r_{1,2} = \lambda \pm i\mu, \quad \lambda, \mu \in \mathbb{R}$ $\qquad i = \sqrt{-1}$ ή $i^2 = -1$

$$\boxed{\text{Ταυτότητα Euler: } e^{ix} = \cos(x) + i\sin(x)}$$

$\Delta < 0 \qquad r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu$

$$y_1^* = e^{r_1 t} = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$$
$$y_2^* = e^{r_2 t} = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{i(-\mu t)} = e^{\lambda t}(\cos(-\mu t) + i\sin(-\mu t)) =$$
$$= e^{\lambda t}(\cos(\mu t) - i\sin(\mu t))$$

$$\bar{z} + z = 2\,\mathrm{Re}\, z$$
$$z - \bar{z} = 2i\,\mathrm{Im}\, z$$

$$z = \alpha + i\beta$$
$$\bar{z} = \alpha - i\beta$$

ορίζουμε

$$\boxed{\begin{aligned} y_1 &= \frac{1}{2}(y_1^* + y_2^*) = e^{\lambda t}\cos(\mu t) \\ y_2 &= \frac{1}{2i}(y_1^* - y_2^*) = e^{\lambda t}\sin(\mu t) \end{aligned}}$$

Γενική λύση: $\boxed{y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)}$

$\alpha y'' + by' + cy = 0$

$y(t_0) = y_0$

$y'(t_0) = y_0'$

$$\boxed{\begin{aligned} y'(t) = {} & c_1 \lambda e^{\lambda t}\cos(\mu t) - c_1 \mu e^{\lambda t}\sin(\mu t) + \\ & + c_2 \lambda e^{\lambda t}\sin(\mu t) + c_2 \mu e^{\lambda t}\cos(\mu t) \end{aligned}}$$

$$\underbrace{\begin{bmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{bmatrix}}_{\Psi} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

Για να υπάρχει μοναδική λύση θέλουμε $\det \Psi \neq 0$

$$\det\varphi = \lambda e^{2\lambda t}\cancel{\cos(\mu t)\sin(\mu t)} + \mu e^{2\lambda t}\cos^2(\mu t) - \lambda e^{2\lambda t}\cancel{\cos(\mu t)\sin(\mu t)} + \mu e^{2\lambda t}\sin^2(\mu t) =$$

$$= \mu e^{2\lambda t}\big(\underbrace{\cos^2(\mu t) + \sin^2(\mu t)}^{=1}\big) = \underset{\neq 0}{\mu}\,\underset{>0}{e^{2\lambda t}} \neq 0.$$

<u>Παράδειγμα</u>:

$$y'' - 4y' + 5y = 0$$

$$y(0) = 0$$

$$y'(0) = 1$$

$$ar^2 + br + c = 0$$

$$\Delta = b^2 - 4ac < 0$$

$$r_{1,2} = \frac{-b \pm i\sqrt{|\Delta|}}{2a}$$

$$r_{1,2} = 2 \pm i \qquad \left(\begin{array}{l}\lambda = 2 \\ \mu = 1\end{array}\right)$$

$$y(t) = c_1 e^{2t}\cos(t) + c_2 e^{2t}\sin(t) \quad \leftarrow \text{Γενική λύση.}$$

$y(0) = c_1 = 0 \Rightarrow y(t) = c_2 e^{2t}\sin(t)$

$y'(t) = 2c_2 e^{2t}\sin(t) + c_2 e^{2t}\cos(t)$

$y'(0) = c_2 = 1$

Άρα η μοναδική λύση του ΠΑΤ είναι η $\boxed{y(t) = e^{2t}\sin(t)}$

Homework $2x + y^2 + 2xyy' = 0$ (Γραμμικές $y'(t) + p(t)y(t) = q(t)$)

Είναι μη γραμμική δ.ε

$M(x,y) + N(x,y)y' = 0$, $M(x,y) = 2x + y^2$, $N(x,y) = 2xy$.

Θέλουμε να εξετάσουμε

$M_y = N_x$

$M_y = 2y$, $N_x = 2y$ άρα η εξίσωση είναι ακριβής.

Άρα $\exists \psi$ τ.ω $\psi_x = M$ και $\psi_y = N$

$$\psi_x = M = 2x + y^2 \Rightarrow \int \psi_x \, dx = \int (2x + y^2)\, dx = \underbrace{2\int x\, dx}_{\cancel{2}\frac{x^2}{\cancel{2}}} + \underbrace{y^2 \int dx}_{y^2 x}$$

$$\Rightarrow \boxed{\psi(x,y) = x^2 + y^2 x + C(y)}$$

$$\psi_y(x,y) = 2xy + C'(y) = N(x,y) = 2xy \Rightarrow C'(y) = 0 \Rightarrow C(y) = c$$

άρα

$$\psi(x,y) = x^2 + y^2 x$$

Γνωρίζουμε ότι η λύση της εξίσωσης δίνεται ως $\psi(x,y) = c$, $c \in \mathbb{R}$

$$\Rightarrow x^2 + y^2 x = c \Rightarrow y^2 = \frac{c - x^2}{x} = \frac{c}{x} - x \Rightarrow y = \pm\sqrt{\frac{c}{x} - x}$$

Άσκηση 3 set 1

$y' - y = t \;,\; y(0) = 1$

$y' = y + t \qquad$ (Μορφή $\; y' = f(y,t)$)

$$y'(t) = f(y,t) \Rightarrow \int_0^t y'(s)\,ds = \int_0^t f(y(s),s)\,ds \Rightarrow$$

$$\Rightarrow y(t) - \overbrace{y(0)}^{1} = \int_0^t f(y(s),s)\,ds$$

$$\Rightarrow y(t) = \int_0^t f(y(s),s)\,ds + 1$$

$\phi_0(t) = 1$ (ικανοποιεί την αρχική συνθήκη)

$$\phi_1(t) = \int_0^t f(\phi_0(s),s)\,ds + 1 = \int_0^t (\cancel{\phi_0(s)}^{\,1} + s)\,ds + 1 =$$

$$= \int_0^t (1+s)\,ds + 1 = t + \frac{t^2}{2} + 1 = \phi_1(t). \qquad \left(f(y,t) = y + t \right)$$

$$\phi_2(t) = \int_0^t \left(s + \frac{s^2}{2} + 1 + s\right) ds + 1 = t^2 + \frac{t^3}{6} + t + 1$$

Άσκηση 18 set 1

$$y' + \frac{1}{t} y = e^{-t^2}, \quad y(1) = e/2.$$

Μορφή $y' + p(t)y = q(t)$

$$y(t) = e^{-\int p(t)dt} \left(\int e^{\int p(t)dt} q(t)\,dt + c \right)$$

$$p(t) = \frac{1}{t}, \quad q(t) = e^{-t^2}$$

$$\int p(t)\,dt = \int \frac{dt}{t} = \ln|t| \overset{\text{αφού } t \geq 1}{=} \ln(t)$$

$$\int e^{\ln t} e^{-t^2} dt = \int t e^{-t^2} dt \ldots =$$

$$\left(e^{-t^2}\right)' = -2te^{-t^2} \Rightarrow te^{-t^2} = -\frac{1}{2}\left(e^{-t^2}\right)'$$

$\dots = -\frac{1}{2}\int \left(e^{-t^2}\right)' dt = -\frac{1}{2} e^{-t^2}$ $\qquad$ $e^{-\ln t} = e^{\ln t^{-1}} = t^{-1} = \frac{1}{t}$

$$\Rightarrow y(t) = e^{-\ln t}\left(-\frac{1}{2} e^{-t^2} + c\right) = \frac{1}{t}\left(c - \frac{1}{2} e^{-t^2}\right)$$

$y(1) = e/2$

$y(1) = c - \frac{1}{2} e^{-1} = e/2 \Rightarrow c = \frac{e + e^{-1}}{2}$

Άρα $\boxed{y(t) = \frac{e + e^{-1} - e^{-t^2}}{2t}}$

Άσκηση 15 set 1

$y' + \frac{3}{t} y = t^4 \ , \ y(1) = 1$ $\qquad$ $p(t) = \frac{3}{t} \ , \quad q(t) = t^4$

$$\int \frac{3}{t}\,dt = 3\ln|t| \overset{t\geq 1}{=} 3\ln t$$

$$\int e^{3\ln t}\, t^4\,dt = \int \cancel{e}^{\cancel{\ln} t^3}\, t^4\,dt = \int t^7\,dt = \frac{t^8}{8}$$

Γενική λύση: $y(t) = e^{-3\ln t}\left(\frac{t^8}{8} + c\right) = \frac{1}{t^3}\left(\frac{t^8}{8} + c\right) = \left(\frac{t^5}{8} + \frac{c}{t^3}\right)$

$$y(1) = \left(\frac{1}{8} + c\right) = 1 \Rightarrow c = \frac{7}{8}$$

Η μοναδική λύση είναι $y(t) = \frac{t^5}{8} + \frac{7}{8t^3}$