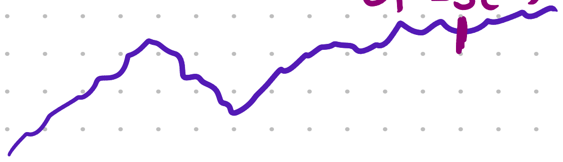
$$\begin{cases} u_t = k u_{xx}, & x \in \mathbb{R},\ t \in [0,T],\ T>0 \\ u(x,0) = f(x), & x \in \mathbb{R} \leftarrow \text{Α.Σ} \end{cases}$$

$2^η$ αναπαράσταση της λύσης

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{+\infty} f(x-\xi)\, e^{-\xi^2/4kt}\, d\xi$$

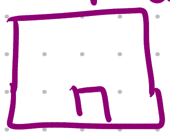
Σύνδεση της εξίσωσης Θερμότητας στο $\mathbb{R}$ με την τιμολόγηση χρηματοοικονομικών παραγώγων

Χρηματοοικονομικό παράγωγο



$S_t = S(t)$ η τιμή μιας μετοχής στο χρόνο t.

S_0 η τιμή της μετοχής σήμερα.



$1€ \longrightarrow 1 \cdot e^{rt}$, $r = 0.05$

1 συμβόλαιο γραμμένο στη μετοχή

$$f(S_T) = (S_T - K)^+,\ T>0 = \begin{cases} S_T - K, & S_T - K > 0 \\ 0, & S_T - K \le 0 \end{cases},\quad K>0$$

$S_0 = 340€$ $\quad T = 1\,\text{year}$ $\quad K$ - strike (Τιμή άσκησης)

$$f(S_T) = (S_T - \underbrace{350}_{K})^+$$

$S_T = 500€ \quad \rightarrow \quad (500-350) = 150€$

$S_T = 300€ \quad \rightarrow \quad (300-350)^+ = 0€$

$$\mathbb{E}\left[e^{-rT} f(S_T) \mid S_0 = 340€ \right] = V(S_0, T)$$

S_0: αρχική τιμή μετοχής

V: δίκαιη τιμή συμβολαίου

T: χρόνος άσκησης συμβολαίου.

↓ Feynman - Kâc

$$\begin{cases} \dfrac{\partial V}{\partial t} + rS\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} - rV = 0 \quad (*) \\ V(S,T) = (S-K)^+ \end{cases}$$

$V = V(S,t)$

$$(S-K)^+ = \begin{cases} S-K, & \text{αν } S > K \\ 0, & \text{αν } S \leq K \end{cases}$$

(Μοντέλο Black - Scholes - Χρήμα 2)

$$t = T - \frac{2\tau}{\sigma^2} \Rightarrow \tau = \frac{\sigma^2}{2}(T-t) \quad , \quad t \in [0,T].$$

$$S = e^x \quad , \quad x = \ln S$$

ορίζω την συνάρτηση

$$V(S,t) = V\left(e^x, T - \frac{2\tau}{\sigma^2}\right) \doteq u(x,\tau) \quad , \quad x \in \mathbb{R} \quad , \quad \tau \in \left[0, \frac{\sigma^2}{2}T\right]$$

$$t=0 \Rightarrow \tau = \frac{\sigma^2}{2}T$$

$$t=T \Rightarrow \tau = 0$$

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\frac{d\tau}{dt} = \frac{\partial u}{\partial \tau}\frac{d\tau}{dt} = -\frac{\sigma^2}{2}\frac{\partial u}{\partial \tau}$$

$$(fg)' = f'g + fg'$$

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\frac{dx}{dS} = \frac{\partial u}{\partial x}\frac{dx}{dS} = \frac{1}{S}\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\text{---}\right) \qquad \frac{\partial\left(\text{---}\right)}{\partial S}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\right)\frac{\partial u}{\partial x} + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial u}{\partial x}\right) =$$

$$= -\frac{1}{s^2}\frac{\partial u}{\partial x} + \frac{1}{s^2}\frac{\partial^2 u}{\partial x^2}$$

Με αντικατάσταση στην (*)

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{2r}{\sigma^2} - 1\right)\dfrac{\partial u}{\partial x} - \dfrac{2r}{\sigma^2} u = 0 \quad (**) \\ u(x,0) = (e^x - k)^+ \end{cases} \qquad u = u(x,\tau)$$

($\frac{2r}{\sigma^2}$: γ)

$$e^{ax+\beta\tau}$$

Αλλαγές μεταβλητών

$$u(x,\tau) \overset{\circ}{=} e^{ax+\beta\tau}\, w(x,\tau) = \Phi(x,\tau)\, w(x,\tau)$$

$$\frac{\partial u}{\partial \tau} = \frac{\partial \Phi}{\partial \tau} w + \Phi \frac{\partial w}{\partial \tau} = \beta \Phi w + \Phi \frac{\partial w}{\partial \tau}$$

$$\frac{\partial u}{\partial x} = \frac{\partial \Phi}{\partial x} w + \Phi \frac{\partial w}{\partial x} = \alpha \Phi w + \Phi \frac{\partial w}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \alpha^2 \Phi w + 2\alpha \Phi \frac{\partial w}{\partial x} + \Phi \frac{\partial^2 w}{\partial x^2}$$

(**) $\Rightarrow$

$$\beta \Phi w + \Phi \frac{\partial w}{\partial \tau} = a^2 \Phi w + 2a\Phi \frac{\partial w}{\partial x} + \Phi \frac{\partial^2 w}{\partial x^2} +$$

$$+ (\nu - 1)\left(a\Phi w + \Phi \frac{\partial w}{\partial x}\right) - \beta \Phi w$$

Όρισα $a = -\frac{1}{2}(\nu-1)$, $\beta = -\frac{1}{4}(\nu+1)^2$

$$\Rightarrow \begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}, & x \in \mathbb{R}, \quad t \in \left[0, \dfrac{\sigma^2}{2} T\right] \\ w(x,0) = e^{-\frac{1}{2}(\nu-1)x} u(x,0) \end{cases}$$

Εξίσωση θερμότητας

Λύνε για $w \longrightarrow u = e^{ax+\beta\tau} w \longrightarrow V$

$s = e^x$

$t = T - \frac{2\tau}{\sigma^2}$